The discrete quantum origin of the Lorentz group and the Z_3-graded ternary algebras (Mathematical aspects of quantum fields and related topics)

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The discrete quantum origin of the Lorentz group and the Z_3-graded ternary algebras

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Abstract

We investigate certain Z_3-graded associative algebras with cubic Z_3 invariant constitutive relations, introduced by one of us some time ago. The invariant forms on finite algebras of this type are given in the cases with two and three generators. We show how the Lorentz symmetry represented by the $SL(2,C)$ group can be introduced without any notion of metric, just as the symmetry of Z_3-graded cubic algebra with two generators, and its constitutive relations. Its representation is found in terms of the Pauli matrices. The relationship of such algebraic constructions with quark states is also considered.

1 Introduction

The great divide between the discrete and the continuum phenomena is one of the most profound dichotomies present since time immemorial not only in mathematics and physics, but also in our global perception of reality. The controversy between Newton and Hyugens concerning the nature of light, or that between the partisans of atomistic theory and those who defended the notion of continuous fluids, between classical thermodynamics and statistical mechanics are the most memorable examples of this everlasting discussions. And of course, the discovery of quanta and quantum physics created new logical traps and difficulties in physics, especially when we try to deduce the laws of quantum physics from its classical limit. Apparently, the opposite point of view, supposing that classical physics of continua is in fact an illusion created by our senses, seems to be more adequate,

The Lorentz and Poincaré groups were established as symmetries of the observable macroscopic world. More precisely, they were conceived in order to take into account the relations between electric and magnetic fields as seen by different Galilean observers. Only later on Einstein extended the Lorentz transformations to space and time coordinates, giving them a universal meaning. As a result, the Lorentz symmetry became perceived as group of invariance of Minkowskian space-time metric. Extending the Lorentz transformations to space and time coordinates modified also Newtonian mechanics so that it could become invariant under the Lorentz instead of the Galilei group.

In the textbooks introducing the Lorentz and Poincaré groups the accent is put on the transformation properties of space and time coordinates, and the invariance of the Minkowskian metric tensor $g_{\mu\nu} = diag(+,-,-,-)$.

But neither the components of $g_{\mu\nu}$, nor the space-time coordinates of an observed event can be given an intrinsic physical meaning; they are not related to any conserved or directly observable quantities. The attempts in order to define a position operator in quantum mechanics have never led to a consistent and unequivocal result ([2], [1]) Under a closer scrutiny, it turns out that only TIME - the proper time of the observer - can be measured directly. The macroscopic notion of space variables results from the convenient description of experiments and observations concerning the propagation of photons, and the existence of the universal constant c.

Consequently, with high enough precision one can infer that the Doppler effect is relativistic, i.e. the frequency ω and the wave vector $\mathbf{k}$ form an entity that is seen differently by different inertial observers, and passing from $\frac{\omega}{c}, \mathbf{k}$ to $\frac{\omega'}{c}, \mathbf{k}'$ is the Lorentz transformation. Another measurable effect leading to the group rules of Lorentz transformations is the aberration of light from stars, (noticed first by Bradley in 1729)

Both effects, proving the relativistic formulae

$$\omega' = \frac{\omega - Vk}{\sqrt{1-\frac{V^2}{c^2}}}, \quad k' = \frac{k - \frac{V}{c^2}\omega}{\sqrt{1-\frac{V^2}{c^2}}},$$

have been checked experimentally by Ives and Stilwell in 1937, although to a limited precision.

Reliable experimental confirmations of the validity of Lorentz transformations concern measurable quantities such as charges, currents, energies (frequencies) and momenta (wave vectors much more than the less intrinsic quantities which are the *differentials* of the space-time variables. In principle, the Lorentz transformations could have been established by very precise observations of the Doppler effect alone. It should be stressed that had we only the light at our disposal, i.e. massless photons propagating with the same velocity c, we would infer that the general symmetry of physical phenomena is the *Conformal Group*, and not the Lorentz-Poincaré group.

To the observations of light must be added the *the principle of inertia*, i.e. the existence of massive bodies moving with velocities lower than c, and supposed constant if not sollicited by external influence.

But to observe a photon, we must capture it with an appropriate device, which may be the retina of our own eye, or any photosensitive device. Upon a closer scrutiny, the observation of a photon is possible only when it interacts with an electron (or another lepton or a quark). Moreover, all photons we observe were emitted by electrons (or leptons or quarks), or bounced from them via Compton scattering. Therefore, it is reasonable to admit that if photons transform according to the vector representation of the Lorentz group, this symmetry property is generated by the symmetry underlying photon-fermion interaction, thus the fundamental symmetry of fermionic states.

At this point it becomes natural to rewrite the combined transformation that acts on fermionic states via an $SL(2, \mathbf{C})$ matrix and on the electric current four-vector via the corresponding 4×4 Lorentz matrix

$$\mid \psi > \rightarrow S \mid \psi > = \mid \psi' >, \quad j^{\mu} = < \psi \mid \gamma^{\mu} \mid \psi > \rightarrow j^{\mu'} = \Lambda^{\mu'}_{\nu}(S)\, j^{\nu}. \tag{1}$$

Our aim is to derive the symmetries of the space-time, i.e. the Lorentz transformations, from the discrete symmetries of the interactions between the most fundamental constituents of matter, in particular quarks and leptons,

We show how the discrete symmetries Z_2 and Z_3 combined with the superposition principle result in the $SL(2, \mathbf{C})$-symmetry. The role of Pauli's exclusion principle in the derivation of the $SL(2, \mathbf{C})$ symmetry is put forward as the source of the macroscopically observed Lorentz symmetry.

2 The quantum origin of the $SL(2, \mathbf{C})$ symmetry

The Pauli exclusion principle ([3]), according to which two electrons cannot be in the same state characterized by identical quantum numbers, is one of the most important cornerstones of quantum physics. This principle not only explains the structure of atoms and therefore the entire content of the periodic table of elements, but it also guarantees the stability of matter preventing its collapse, as suggested by Ehrenfest ([4]), and proved later by Dyson ([5], [6]). The relationship between the exclusion principle and particle's spin, known under the name of the "spin-and-statistic theorem", represents one of the deepest results in quantum field theory.

In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states. The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the "bra" and "ket" vectors.

Consider the probability amplitude to find a particle in the state $\mid 1 >$,

$$\Phi(1) = < \psi \mid 1 > . \tag{2}$$

The wave function of a two-particle state of which one is in the state $\mid 1>$ and another in the state $\mid 2>$ (all other observables supposed to be the same for both states) is represented by a superposition

$$\mid \psi >= \Phi(1,2)\,(\mid 1> \otimes \mid 2>). \tag{3}$$

It is clear that if the wave function $\Phi(1,2)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(1,2) = -\Phi(2,1), \tag{4}$$

then $\Phi(1,1) = 0$ and such states have vanishing both their wave function and probability. It is easy to prove using the superposition principle, that this condition is not only sufficient, but also necessary.

Let us suppose that $\Phi(i,k)$ $(i,k = 1,2)$ does vanish when $i = k$. This remains valid in any basis provided the new basis $\mid 1' >, \mid 2' >$ was obtained from the former one via a unitary transformation. Let us form an arbitrary state being a linear combination of $\mid 1>$ and $\mid 2>$,

$$\mid z >= \alpha \mid 1> + \beta \mid 2>, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z,z) =< \psi \mid (\alpha \mid 1> + \beta \mid 2>) \otimes (\alpha \mid 1> + \beta \mid 2>), \tag{5}$$

which develops as follows:

$$\alpha^2 < \psi \mid (1,1) > + \alpha\beta < \psi \mid (1,2) > + \beta\alpha < \psi \mid (2,1) > + \beta^2 < \psi \mid (2,2) >=$$

$$= \Phi(z,z) = \alpha^2\,\Phi(1,1) + \alpha\beta\,\Phi(1,2) + \beta\alpha\,\Phi(2,1) + \beta^2\,\Phi(2,2). \tag{6}$$

Now, as $\Phi(1,1) = 0$ and $\Phi(2,2) = 0$, the sum of remaining two terms will vanish if and only if (4) is satisfied, i.e. if $\Phi(1,2)$ is anti-symmetric in its two arguments.

After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations

$$a^\dagger(1)a^\dagger(2) + a^\dagger(2)a^\dagger(1) = 0, \quad a(1)a(2) + a(2)a(1) = 0 \tag{7}$$

The bottom line is that the Hilbert space of fermionic states is always divided in two sectors corresponding to the anti-commutation of creation of dichotomic spin state, admitting only two values which are labeled $+\frac{1}{2}$ and $+\frac{1}{2}$. The anti-commuting character of their operator algebra of observables is represented by the antisymmetric tensor $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\alpha, \beta = 1,2$. The exclusion principle being universal, it is natural to require that it should be independent of the choice of a basis in the Hilbert space of states. Therefore, if the states undergo a linear transformation

$$\mid \psi^\alpha > \rightarrow \mid \psi_{\beta'} >= S^\alpha_{\beta'} \mid \psi_\alpha >, \tag{8}$$

the anti-symmetric form which encodes the exsclusion principle, should remain the same as before, thus

$$\epsilon_{\alpha'\beta'} = S^{\alpha}_{\alpha'} S^{\beta}_{\beta'} \epsilon_{\alpha\beta} \tag{9}$$

with

$$\epsilon_{1'2'} = -\epsilon_{2'1'} = 1, \quad \epsilon_{1'1'} = 0, \quad \epsilon_{2'2('} = 0. \tag{10}$$

This invariance condition, akin to the invariance of the metric tensor $\eta_{\mu\nu}$ of the Minkowskian spacetime, defines the invariance group. It is easy to see that in this case the combined effect of (9) and (10) lead to the definition of the $SL(2, \mathbf{C})$ group. It is enough to check one of the four equations (9), e.g. choosing $\alpha' = 1, \ \beta' = 2$. We get then

$$\epsilon_{1'2'} = 1 = S^{\alpha}_{1'} S^{\beta}_{2'} \, \epsilon_{\alpha\beta} = S^{1}_{1'} S^{2}_{2'} - S^{2}_{1'} S^{1} 2' = \det S = 1. \tag{11}$$

The other three choices of index values in (9) are either redundant, or trivial, i.e. leading to the identity $0 = 0$.). The conjugate matrices span an inequivalent representation of the $SL(2, \mathbf{C})$ group, labeled by dotted indeces; the invariant antisymmetric 2-form leads to the same result when its invariance is required:

$$\epsilon_{\dot{1}\dot{2}} = -\epsilon_{\dot{2}\dot{1}} = 1, \quad \epsilon_{\dot{1}\dot{1}} = 0, \quad \epsilon_{\dot{2}\dot{2}} = 0; \quad \epsilon_{\dot{1}'\dot{2}'} = \bar{S}^{\dot{\alpha}}_{\dot{\alpha}'} \bar{S}^{\dot{\beta}}_{\dot{2}'} \, \epsilon_{\dot{\alpha}\dot{\beta}} \tag{12}$$

3 A Z_3-graded ternary generalization of fermions

Consider an associative algebra $\mathcal{A}$ over $\mathbf{C}^1$ spanned by N generators $\theta^A, \ \ A, B, ... = 1, 2, ..., N$. The generators θ^A are given the grade 1; their N^2 linearly independent binary poducts $_t hat^A \theta^B$ are of grade 2, whereas their cubic products of grade $3 = 0_{mod\, 3}$ are subject to the following *cubic commutation relations*:

$$\theta^A \theta^B \theta^C = j \, \theta^B \theta^C \theta^A = j^2 \, \theta^C \theta^A \theta^B, \qquad \text{with } j = e^{\frac{2\pi i}{3}}, \tag{13}$$

Obviously, due to the associativity property, all higher-order monomials starting from order 4 do vanish automatically; the proof is by direct calculus.

A *conjugate algebra* of the same dimension, $\bar{\mathcal{A}}$ is introduced, with N conjugate generators $\bar{\theta}^{\dot{A}}$ of grade 2, their quadratic products of grade 1 being spanned by the expressions $\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}$, while their cubic products satisfy conjugate ternary commutation rules,

$$\bar{\theta}^{\dot{A}} \, \bar{\theta}^{\dot{B}} \, \bar{\theta}^{\dot{C}} = j^2 \, \bar{\theta}^{\dot{A}} \, \bar{\theta}^{\dot{B}} \, \bar{\theta}^{\dot{C}}$$

The two algebras can be united into one common structure if we define new relations between the θ^A and $\bar{\theta}^{\dot{B}}$ generators. We propose the following choice:

$$\theta^A \bar{\theta}^{\dot{B}} = -j \bar{\theta}^{\dot{B}} \theta^A, \quad \bar{\theta}^{\dot{B}} \theta^A = -j^2 \theta^A \bar{\theta}^{\dot{B}}, \tag{14}$$

because they lead to the anti-commutation between the ternary products:

$$(\theta^A \theta^B \theta^C)(\bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}}) = -(\bar{\theta}^{\dot{D}} \bar{\theta}^{\dot{E}} \bar{\theta}^{\dot{F}})(\theta^A \theta^B \theta^C) \tag{15}$$

The invariant 3-forms are defined as follows:

$$\rho^{\alpha}_{ABC}\theta^A\theta^B\theta^C = \rho^{\alpha}_{BCA}\theta^B\theta^C\theta^A = \rho^{\alpha}_{CAB}\theta^C\theta^A\theta^B; \tag{16}$$

But this means that we must have

$$\rho^{\alpha}_{ABC} = j\,\rho^{\alpha}_{BCA} = j^2\,\rho^{\alpha}_{CAB}. \tag{17}$$

The upper index α runs from 1 to $(N^3 - N)/3$ when the algebra $\mathcal{A}$ is spanned by N generators.

The conjugate 3-forms $\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}}$ satisfy a similar symmetry conditions,

$$\bar{\rho}^{\dot{\alpha}}_{\dot{A}\dot{B}\dot{C}} = j^2\,\bar{\rho}^{\dot{\alpha}}_{\dot{B}\dot{C}\dot{A}} = j\,\bar{\rho}^{\dot{\alpha}}_{\dot{C}\dot{A}\dot{B}}. \tag{18}$$

Let us concentrate our investigation on the two-dimensional case. From now on, we shall admit that only two values are taken by the indeces $A, B, ...$ as well as by the dotted ones, $\dot{C}, \dot{D}$. In this case, the indeces $\alpha, \beta, ...$ and $\dot{\gamma}, delta, ...$ also run from 1 to 2. The ρ^{α}_{ABC}-matrices can be then normalized in the following way:

$$\rho^1_{121} = 1, \quad \rho^1_{211} = j^2, \quad \rho^1_{112} = j, \quad \text{other components } = 0,$$
$$\rho^2_{212} = 1, \quad \rho^2_{122} = j^2, \quad \rho^2_{221} = j, \quad \text{other components } = 0, \tag{19}$$

The conjugate 3-forms are defined in an obvious way:

$$\bar{\rho}^{\dot{1}}_{\dot{1}\dot{2}\dot{1}} = 1, \quad \bar{\rho}^{\dot{1}}_{\dot{2}\dot{1}\dot{1}} = j, \quad \bar{\rho}^{\dot{1}}_{\dot{1}\dot{1}\dot{2}} = j^2, \quad \text{other components } = 0,$$
$$\bar{\rho}^{\dot{2}}_{\dot{2}\dot{1}\dot{2}} = 1, \quad \bar{\rho}^{\dot{2}}_{\dot{1}\dot{2}\dot{2}} = j, \quad \bar{\rho}^{\dot{2}}_{\dot{2}\dot{2}\dot{1}} = j^2, \quad \text{other components } = 0 \tag{20}$$

Similarly, invariant two-forms can be introduced, satisfying the following relation:

$$\pi^{\mu}_{A\dot{B}}\,\theta^A\bar{\theta}^{\dot{B}} = \bar{\pi}^{\mu}_{\dot{B}A}\,\bar{\theta}^{\dot{B}}\theta^A. \tag{21}$$

The index $\mu, \nu ...$ takes on *four* different values, which we can label symbolically $0, 1, 2, 3$, correspondiong to four different independent combinations of dotted and un-dotted indeces $\dot{B}$ and A: $1\dot{1}$, $1\dot{2}$, $2\dot{1}$ and $2\dot{2}$ It is easy to check that this means that the matrices $\pi^{\mu}_{A\dot{B}}$ should satisfy the relation

$$\pi^{\mu}_{A\dot{B}} = -j^2\bar{\pi}^{\mu}_{\dot{B}A}, \tag{22}$$

Four 2×2 matrices satisfying (21) are easily found to be given by the Pauli matrices with an appropriate factors:

$$\pi^{\mu}_{A\dot{B}} = i\,j\,\sigma^{\mu}_{A\dot{B}}, \qquad \bar{\pi}^{\mu}_{\dot{B}A} = -i\,j^2\,\bar{\sigma}^{\mu}_{\dot{B}A}, \tag{23}$$

because the matrices $\bar{\sigma}^{\mu}_{A\dot{B}}$ are hermitian, so that

$$\sigma^{\mu}_{A\dot{B}} = \bar{\sigma}^{\mu}_{\dot{B}A}.$$

It is worthwhile to note that by introducing also the minus sign, in other woirds, by multiplying by $-j$ we in fact enlarged the symmetry from Z_3 to $Z_2 \times Z_3 = Z_6$. The latter can be generated by one of the non-trivial sixth-order roots of unity, e.g. by $-j$.

4 Further investigation of Z_3-graded ternary algebra

Let us first fix the convention for raising and lowering various indeces. We wish to introduce the covariant basis θ_A, $\bar{\theta}_{\dot{B}}$,, satisfying similar cubic relations. With two generators only the choice of a covariant 2-tensor T_{AB} that would serve to lower the contravariant indeces of the generators θ^A is quite limited. As a matter of fact, it can be always reduced by an appropriate linear transformation of the basis, to either the symmetric one, the Kronecker delta δ_{AB}, or to the anti-symmetric two-form ϵ_{AB}.

We suppose that the upper indeces α, β and $\dot{\gamma}$, $\dot{\delta}$ belong to the usual bi-spinors which taken together span the space of spinorial representation of the Lorentz group, acting via its double covering by $SL(2, \mathbf{C})$ as follows:

$$\phi^{\alpha'} = S^{\alpha'}_{\beta}\, \phi^{\beta}, \quad \chi^{\dot{\alpha}'} = \bar{S}^{\dot{\alpha}'}_{\dot{\beta}}\, \chi^{\dot{\beta}}. \tag{24}$$

This hypothesis is justified by the fact that composite particles (proton, neutron, etc.) containing *three* quarks behave like Lorentz spinors; we believe that they can be constructed as cubic combinations of quarks. In the realm of first quantization this would mean that their wave functions can be produced by cubic products of wave functions of quarks; in the realm of second quantization we should aim at constructing the appropriate fermionic creation and annihilation operators as cubic products of creation or annihilation operators of single quark states.

If we want to keep the transformation property independent of the choice of basis, then the invariant tensor with respect to the action of $SL(2, \mathbf{C})$ group is the anti-symmetric two-form and its contravariant inverse:

$$\epsilon_{12} = 1, \;\; \epsilon_{21} = -1, \;\; \epsilon_{11} = 0, \;\; \epsilon_{22} = 0;$$
$$\epsilon^{12} = 1, \;\; \epsilon^{21} = -1, \;\; \epsilon^{11} = 0, \;\; \epsilon^{22} = 0. \tag{25}$$

Then, if we want to make the contravariant counterparts of our cubic matrices ρ^{ABC}_{α} satisfy the same definition as the original forms ρ^{α}_{ABC}, we must raise the quark indeces $A, B...$ by means of the same anti-symmetric tensor ϵ^{AB}. Then one easily checks that the contravariant tensors ρ^{ABC}_{α} defined as follows:

$$\rho^{ABC}_{\alpha} = \epsilon_{\alpha\beta}\, \rho^{\beta}_{DEF}\, \epsilon^{AD}\, \epsilon^{BE}\, \epsilon^{CF} \tag{26}$$

have the same components as the covariant ones,

$$\rho^{121}_{1} = 1, \;\; \rho^{211}_{1} = j^2, \;\; \rho^{112}_{1} = j, \qquad \rho^{212}_{2} = 1, \;\; \rho^{122}_{2} = j^2, \;\; \rho^{221}_{2} = j.$$

Similar properties are displayed by the conjugate entities with dotted indeces, raised and lowered by the dotted anti-symmetric tensors $\epsilon_{\dot{\alpha}\dot{\beta}}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$ and $\epsilon_{\dot{A}\dot{B}}$, $\epsilon^{\dot{A}\dot{B}}$ It tuns out that certain representation of $SL(2, \mathbf{C})$ leaves these three-forms invariant: as a matter of fact, one has

$$S^{\alpha'}_{\beta}\, \rho^{\beta}_{ABC} = \rho^{\alpha'}_{A'B'C'}\, U^{A'}_{A}\, U^{B'}_{B}\, U^{C'}_{C} \tag{27}$$

where $S^{\alpha'}_{\beta}$ are the usual complex 2×2 matrices of the basic representation of the $SL(2,\mathbf{C})$ group, whereas the 2×2 complex matrices $U^{A'}_{A}$ are defined as follows:

$$U^{1'}_{1} = S^{1'}_{1}\det(U),\quad U^{1'}_{2} = -S^{1'}_{2}\det(U),\quad U^{2'}_{1} = -S^{2'}_{1}\det(U),\quad U^{2'}_{2} = S^{2'}_{2}\det(U). \qquad (28)$$

so that one has

$$\det(S) = [\det(U)]^3 . \qquad (29)$$

The same is true for the conugate matrices $\bar{U}^{\dot{A}}_{\dot{B}}$: one has also $\det(S) = \left[\det(\bar{U})\right]^3$.

As $\det(S) = 1$, the determinant of U can be equal to 1, or j, or j^2. Let us choose $\det U = j$ and $\det(\bar{U}) = j^2$.

Using the invariant 2-forms ϵ^{AB} and $\epsilon\dot{C}\dot{D}$ for raising and contracting indeces, we can construct a symmetric tensor $g^{\mu\nu}$, $\mu,\nu,.. = 0,1,2,3$ as follows:

$$g^{\mu\nu} = \pi^{\mu}_{A\dot{B}}\,\bar{\pi}^{\nu}_{\dot{D}C}\,\epsilon^{AB}\epsilon^{\dot{C}\dot{D}}. \qquad (30)$$

whose components define the Minkowskian space-time metric:

$$g^{00} = 1,\quad g^{11} = g^{22} = g^{33} = -1,\quad g^{mu\nu} = 0 \text{ if } \mu \neq \nu. \qquad (31)$$

The covariant tensor $g_{\lambda\rho}$ with exactly the same components is uniquely defined by the condition

$$g_{\mu\nu}\,g^{\nu\lambda} = \delta^{\mu}_{\lambda}.$$

The group of invariance of thus defined Minkowskian metric is the Lorentz group transforming the metric tensor so that the components in the new basis,

$$g^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}\,g^{\mu\nu}, \qquad (32)$$

are given by $g^{\mu'\nu'} = \mathrm{diag}(+1,-1,-1,-1)$, i.e. exactly the same components as $g^{\mu\nu}$ in the original basis.

Now, despite the fact that the matrices $\pi^{\mu}_{A\dot{B}}$ and $\bar{\pi}^{\nu}_{\dot{B}A}$ are endowed with slamm Greek indeces μ, they do not transform covariantly under the Lorentz group represented by the set of matrices $\Lambda^{\mu'}_{\mu}$; in fact, as it is quite easy to check,

$$\Lambda^{\mu'}_{\mu}\,\pi^{\mu}_{A\dot{B}} \neq \pi^{\mu'}_{A'\dot{B}'}\,U^{A'}_{A}U^{\dot{B}'}_{\dot{B}},$$

one of the reasons being the fact that with the above transformation we create linear combinations of traceless matrices $\pi^{k}_{A\dot{B}}$ with the unit 2×2 matrix $\pi^{0}_{A\dot{B}}$ whose trace does not vanish (it is equal to 2).

To implement the Lorentz transformations on the matrices $\pi^{\mu}_{A\dot{B}}$ and $\bar{\pi}^{\nu}_{\dot{B}A}$, we have to organize them in 4×4 matrices by analogy with the usual Dirac matrices as follows:

$$\Pi^0 = \begin{pmatrix} \pi^0 & 0 \\ 0 & -\bar{\pi}^0 \end{pmatrix},\ \Pi^1 = \begin{pmatrix} 0 & \pi^1 \\ -\bar{\pi}^1 & 0 \end{pmatrix},\ \Pi^2 = \begin{pmatrix} 0 & \pi^2 \\ -\bar{\pi}^2 & 0 \end{pmatrix},\ \Pi^3 = \begin{pmatrix} 0 & \pi^3 \\ -\bar{\pi}^0 & 0 \end{pmatrix} \qquad (33)$$

Then, as in the case of Dirac's gamma-matrices, we shall have the following covariance property:

$$\mathcal{U}^{-1}\,\Pi^{\mu}\,\mathcal{U} = \Lambda^{\mu}_{\nu}\,\Pi^{\nu},\quad \text{with}\quad \mathcal{U} = \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}. \qquad (34)$$

5 Invariance group of ternary Clifford algebra

Let us introduce the following three 3×3 matrices:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{35}$$

and their hermitian conjugates

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{36}$$

These matrices can be allowed natural Z_3 grading,

$$\mathrm{grade}(Q_k) = 1, \quad \mathrm{grade}(Q_k^\dagger) = 2, \tag{37}$$

The above matrices span a very interesting ternary algebra. Out of three independent Z_3-graded ternary combinations, only one leads to a non-vanishing result. One can check without much effort that both j and j^2 skew ternary commutators do vanish:

$$\{Q_1, Q_2, Q_3\}_j = Q_1 Q_2 Q_3 + j Q_2 Q_3 Q_1 + j^2 Q_3 Q_1 Q_2 = 0,$$

$$\{Q_1, Q_2, Q_3\}_{j^2} = Q_1 Q_2 Q_3 + j^2 Q_2 Q_3 Q_1 + j Q_3 Q_1 Q_2 = 0,$$

and similarly for the odd permutation, $Q_2 Q_1 Q_3$. On the contrary, the totally symmetric combination does not vanish; it is proportional to the 3×3 identity matrix $\mathbf{1}$:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = \eta_{abc} \, \mathbf{1}, \quad a, b, ... = 1, 2, 3. \tag{38}$$

with η_{abc} given by the following non-zero components:

$$\eta_{111} = \eta_{222} = \eta_{333} = 1, \quad \eta_{123} = \eta_{231} = \eta_{312} = 1, \quad \eta_{213} = \eta_{321} = \eta_{132} = j^2. \tag{39}$$

all other components vanishing. The relation 38) may serve as the definition of *ternary Clifford algebra.*

Another set of three matrices is formed by the hermitian conjugates of Q_a, which we shall endow with dotted indeces $\dot{a}, \dot{b}, ... = 1, 2, 3$:

$$Q_{\dot{a}} = Q_a^\dagger \tag{40}$$

satisfying conjugate identities

$$Q_{\dot{a}} Q_{\dot{b}} Q_{\dot{c}} + Q_{\dot{b}} Q_{\dot{c}} Q_{\dot{a}} + Q_{\dot{c}} Q_{\dot{a}} Q_{\dot{b}} = \eta_{\dot{a}\dot{b}\dot{c}} \, \mathbf{1}, \quad \dot{a}, \dot{b}, ... = 1, 2, 3. \tag{41}$$

with $\eta_{\dot{a}\dot{b}\dot{c}} = \bar{\eta}_{abc}$.

It is obvious that any similarity transformation of the generators Q_a will keep the ternary anti-commutator (39) invariant. As a matter of fact, if we define $\tilde{Q}_b = P^{-1} Q_b P$, with P a non-singular 3×3 matrix, the new set of generators will satisfy the same ternary relations, because

$$\tilde{Q}_a \tilde{Q}_b \tilde{Q}_c = P^{-1} Q_a P P^{-1} Q_b P P^{-1} Q_c P = P^{-1} (Q_a Q_b Q_c) P,$$

and on the right-hand side we have the unit matrix which commutes with all other matrices, so that $P^{-1}\, \mathbf{1}\, P = \mathbf{1}$.

However, the change of the basis in our algebra is less trivial, and one may ask the question whether linear transformations of the type

$$Q_{b'} = M^a_{b'}\, Q_a, \quad \text{so that} \quad \eta_{d'f'g'} = M^a_{d'} M^b_{f'} M^c_{g'}\, \eta_{abc} \tag{42}$$

can keep the three-form η invariant, i.e. having exactly the same components as defined by (39) ?

To find out the structure of the group of matrices M leaving the form η invariant, it is enough to investigate its Lie algebra by considering only matrices infinitesimally close to the unit matrix. Therefore, let

$$M^a_{b'} = \delta^a_{b'} + \varepsilon\, L^a_{b'}. \tag{43}$$

Inserting the above matrix into the formula (42),we get the following condition:

$$\eta_{a'b'c'} = \left(\delta^a_{a'} + \epsilon L^a_{a'}\right) \left(\delta^b_{b'} + \epsilon L^b_{b'}\right) \left(\delta^c_{c'} + \epsilon L^c_{c'}\right)\, \eta_{abc}. \tag{44}$$

Developing the product (44) and keeping only the termls linear in ϵ one gets the following eqation:

$$\eta_{a'bc'} L^b_{b'} + \eta_{a'b'c} L^c_{c'} + \eta_{ab'c'} L^a_{a'} = 0. \tag{45}$$

The equations (45) impose a number of conditions on admissible matrices $L^a_{a'}$, which should be solved one by one, choosing all possible sets of lower case indeces $(a'b'c')$. The choice of $(a'b'c') = (111)$ yields $\eta_{111} L^1_1 = 0$, whence $L^1_1 = 0$; similarly, the remaining two diagonal terms vanish, too: $L^2_2 = 0$, $L^3_3 = 0$, which means that matrices keeping the form η invariant are not only traceless, but have only zeros on their diagonal.

Among the remaining choices of three indeces, the components of η_{abc} with *all indeces different*, i.e. essentially only two independent ones, (123) and (213) do not impose any new conditions, becaquse they contain only the diagonal entries of $L^a_{a'}$ which are already set to 0, e.g.:

$$\eta_{123} L^1_1 + \eta_{123} L^2_2 + \eta_{123} M^3_3 = 0,$$

and the same for the odd permutation, η_{213}.

What remains now are the six independent choices of three indeces with two identical and one different:

$$(112), \ \ (113), \ \ (221), \ \ (223), \ \ (331), \ \ (332).$$

The combinations (112), (223) and (331) lead to the following identities:

$$L^1_{2'} = jL^3_{1'}, \;\; jL^1_{2'} = L^2_{3'}, \;\; L^3_{1'} = jL^2_{3'}. \tag{46}$$

while the three remaining choices, (221), (332) and (113) yield another set of conditions,

$$jL^3_{2'} = L^2_{1'}, \;\; L^3_{2'} = jL^1_{3'}, \;\; jL^2_{1'} = M^1_{3'}. \tag{47}$$

This means that all such matrices depend on two real parameters. We can choose $L^1_{2'} = 1$, then we shall have $L^2_{3'} = j$ and $L^3_{1'} = j^2$. The second independent choice is, say, $L^3_{2'} = 1$, then we shall have $L^1_{3'} = j$ and $L^2_{1'} = j^2$. The most general form of matrix conserving the 3-form η_{abc} is thus the following matrix function of two parameters r and s:

$$L(r,s) = r\, L_{(1)} + s\, L_{(2)} = r \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix} + s \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{48}$$

The matrices $L_{(1)}$ and $L_{(2)}$ satisfy the following j-graded commutation relations:

$$L_{(1)}L_{(2)} = j^2\, L_{(2)}L_{(1)}, \quad L_{(2)}L_{(1)} = j\, L_{(1)}L_{(2)}. \tag{49}$$

Finite transformations keeping the form η_{abc} invariant can be now obtained by exponentiation:

$$e^{r\,L_{(1)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix} + \frac{r^2}{2!} \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix} + \ldots$$

$$e^{s\,L_{(2)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + s \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \frac{s^2}{2!} \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix} + \ldots$$

The consecutive powers of generators $L_{(1)}$ and $L_{(2)}$ repeat themselves, so that there are only three different matrices present after the exponentiation. The result can be thus written as

$$e^{rL_{(1)}} = \sum_{n=0}^{\infty} \frac{r^{3n}}{(3n)!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{r^{3n+1}}{(3n+1)!} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{r^{3n+2}}{(3n+2)!} \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}$$

for the first generator, and

$$e^{sL_{(2)}} = \sum_{n=0}^{\infty} \frac{s^{3n}}{(3n)!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{s^{3n+1}}{(3n+1)!} \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{s^{3n+2}}{(3n+2)!} \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}$$

for the second one. In principle, one could find a general transformation by developing into an infinite series the expression

$$e^{rL_{(1)}+sL_{(2)}}$$

using the j-commutation relation (49), which would amount to some generalizaton of the Baker-Campbell-Hausdorff formula for exponentiation of a sum of two non-commuting operators.

In the spirit of search of general covariance of algebras we could pose the problem differently: without demanding the invariance of ternary multiplication table implemented by the 3-form η_{abc}, we could ask a similar question concerning the matrices Q_a themselves. The analogy with the Clifford algebra spanned by the Dirac matrices is quite obvious: we should be looking for matrices S and M such that

$$S^{A}_{A'}\,(Q_{a'})^{A'}_{B'}\,\overset{-1}{S}{}^{B'}_{B} = M^{a}_{a'}(Q_a)^{A}_{B}. \tag{50}$$

As in the previous case, it is enough to investigate the infinitesimal transformations, assuming that our matrices are close to the identity matrix:

$$S^{A}_{A'} \simeq \delta^{A}_{A'} + \epsilon\, W^{A}_{A'}, \quad \overset{-1}{S}{}^{B'}_{B} \simeq \delta^{B'}_{B} + \epsilon\, W^{B'}_{B}, \quad M^{a}_{a'} \simeq \delta^{a}_{a'} + \epsilon\, \Lambda^{a}_{a'} \tag{51}$$

then the condition (50) implies, up to the terms linear in small parameter ϵ, the following identity:

$$[W, Q^a] = WQ^a - Q^a W = \Lambda^a_b Q^b \tag{52}$$

Under matrix multiplication, the Z_3 grades add up modulo three; therefore if we want the commutators in (52) to yield a combination of matrices Q^a of Z_3 grade 1, therefore the matrix W must be of Z_3 grade 0, i.e. *diagonal* in the chosen representation. The basis of 3×3 diagonal matrices contains the unit matrix, which commutes with all other matrices and would not contribute to the commutators in (52), and two traceless matrices

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B_2 = B_1^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \tag{53}$$

Inserting these two matrices in the equation (52), we get the two following matrices Λ^a_b:

$$[B_1, Q^a] = \Lambda_{1}{}^{a}_{b} Q^b, \quad [B_2, Q^a] = \Lambda_{2}{}^{a}_{b} Q^b, \tag{54}$$

$$\text{with } \Lambda_1 = (j - j^2) \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \Lambda_1^\dagger.$$

By exponentiating, we get again a two-parameter transformation group as before.

Similar transformations concern the harmitian conjugates $Q_a^\dagger = Q_{\dot{a}}$, with matrices B_1 and B_2 interchanged. The eight traceless 3×3 matrices

$$Q_1, Q_2, Q_3, Q_1^\dagger, Q_2^\dagger, Q_3^\dagger, B_1, B_2$$

span a Lie algebra with respect to an ordinary commutators, and a Z_3 graded ternary algebra with respect to Z_3-graded cubic commutators

$$\{A, B, C\} := ABC + j^{\alpha\beta\gamma} BCA + j^{2\alpha\beta\gamma} CAB,$$

with $\alpha = grad(A)$, $\beta = grad(B)$, $\gamma = grad(C)$.. The generators B_1, B_2 form a Cartan subalgebra of the Lie algebra spanned by the eight generators, which can be expressed as linear combinations of the Gell-Mann matrices spanning the Lie algebra of the $SU(3)$ group.

The full group of invariance can be recovered if we let all the eight generators act by commutating on themselves, leading to unrestricted linear combnations oa eight generators again. This will define the 8×8 representation of the $SU(3)$ algebra.

Full 8×8 multiplication table of "nonions" can be found e.g. in ([18]).

6 A Z_3-graded generalization of the Dirac equation

The Z_2 symmetry constitutes the essential ingredient in the formulation of Lorentz-invariant equations of relativistic quantum physics. It has a fundamental role in the definition of time reversal and particle-antiparticle symmetry ([17])

Let us first underline the Z_2 symmetry of Maxwell and Dirac equations, which implies their hyperbolic character, which makes the propagation possible. Maxwell's equations *in vacuo* can be written as follows:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \nabla \wedge \mathbf{B}, \quad -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}. \tag{55}$$

These equations can be decoupled by applying the time derivation twice, which in vacuum, where $div\mathbf{E} = 0$ and $div\mathbf{B} = 0$ leads to the d'Alembert equation for both components separately:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \quad \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0.$$

Nevertheless, neither of the components of the Maxwell tensor, be it $\mathbf{E}$ or $\mathbf{B}$, can propagate separately alone. It is also remarkable that although each of the fields $\mathbf{E}$ and $\mathbf{B}$ satisfies a second-order propagation equation, due to the coupled system (55) there exists a quadratic combination satisfying the forst-order equation, the Poynting four-vector:

$$P^\mu = \left[P^0, \mathbf{P}\right], \quad P^0 = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \tag{56}$$

$$\text{with } \mathbf{P} = \mathbf{E} \wedge \mathbf{B}, \text{ with } \partial_\mu P^\mu = 0. \tag{57}$$

The Dirac equation for the electron displays a similar Z_2 symmetry, with two coupled equations which can be put in the following form:

$$i\hbar\frac{\partial}{\partial t}\psi_+ - mc^2\psi_+ = i\hbar\sigma \cdot \nabla\psi_-, \quad -i\hbar\frac{\partial}{\partial t}\psi_- - mc^2\psi_- = -i\hbar\sigma \cdot \nabla\psi_+, \tag{58}$$

where ψ_+ and ψ_- are the positive and negative energy components of the Dirac equation; this is visible even better in the momentum representation:

$$\left[E - mc^2\right] \psi_+ = c\sigma \cdot \mathbf{p}\psi_-, \quad \left[-E - mc^2\right] \psi_- = -c\sigma \cdot \mathbf{p}\psi_+. \tag{59}$$

The same effect (negative energy states can be obtained by changing the direction of time, and putting the minus sign in front of the time derivative, as suggested by Feynman.

Each of the components satisfies the Klein-Gordon equation, obtained by successive application of the two operators and diagonalization:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - m^2\right] \psi_{\pm} = 0$$

As in the electromagnetic case, neither of the components of this complex entity can propagate by itself; only all the components can.

Apparently, the two types of quarks, u and d, cannot propagate freely, but can form a freely propagating particle perceived as a fermion, only under an extra condition: they must belong to three different species called *colors*; short of this they will not form a propagating entity.

Therefore, quarks should be described by *three fields* satisfying a set of coupled linear equations, with the Z_3-symmetry playing a similar role of the Z_2-symmetry in the case of Maxwell's and Dirac's equations. Instead of the "-" sign multiplying the time derivative, we should use the cubic root of unity j and its complex conjugate j^2 according to the following scheme:

$$\frac{\partial \mid \psi >}{\partial t} = \hat{H}_{12} \mid \phi >, \quad j\frac{\partial \mid \phi >}{\partial t} = \hat{H}_{23} \mid \chi >, \quad j^2\frac{\partial \mid \chi >}{\partial t} = \hat{H}_{31} \mid \psi >, \tag{60}$$

We do not specify yet the number of components in each state vector, nor the character of the hamiltonian operators on the right-hand side; the three fields $\mid \psi >$, $\mid \phi >$ and $\mid \chi >$ should represent the three colors, none of which can propagate by itself.

The quarks being endowed with mass, we can suppose that one of the main terms in the hamiltonians is the mass operator $\hat{m}$; and let us suppose that the remaining parts are the same in all three hamiltonians. This will lead to the following three equations:

$$\frac{\partial \mid \psi >}{\partial t} - \hat{m} \mid \psi >= \hat{H} \mid \phi >, \quad j\frac{\partial \mid \phi >}{\partial t} - \hat{m} \mid \phi >= \hat{H} \mid \chi >,$$

$$j^2\frac{\partial \mid \chi >}{\partial t} - \hat{m} \mid \chi >= \hat{H} \mid \psi >, \tag{61}$$

Supposing that the mass operator commutes with time derivation, by applying three times the left-hand side operators, each of the components satisfies the same common *third order* equation:

$$\left[\frac{\partial^3}{\partial t^3} - \hat{m}^3\right] \mid \psi >= \hat{H}^3 \mid \psi > . \tag{62}$$

The anti-quarks should satisfy a similar equation with the negative sign for the Hamiltonian operator. The fact that there exist two types of quarks in each nucleon suggests that the state vectors $\mid \psi >, \mid \phi >$ and $\mid \chi >$ should have two components each. When combined together, the two postulates lead to the conclusion that we must have three two-component functions and their three conjugates:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\bar{\psi}_{\dot{1}}\\ \bar{\psi}_{\dot{2}}\end{pmatrix}, \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \begin{pmatrix}\bar{\varphi}_{\dot{1}}\\ \bar{\varphi}_{\dot{2}}\end{pmatrix}, \quad \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \begin{pmatrix}\bar{\chi}_{\dot{1}}\\ \bar{\chi}_{\dot{2}}\end{pmatrix},$$

which may represent three colors, two quark states (e.g. "up" and "down"), and two anti-quark states (with anti-colors, respectively).

Finally, in order to be able to implement the action of the $SL(2, \mathbf{C})$ group via its 2×2 matrix representation defined in the previous section, we choose the Hamiltonian $\hat{H}$ equal to the operator $\sigma \cdot \nabla$, the same as in the usual Dirac equation. The action of the Z_3 symmetry is represented by factors j and j^2, while the Z_2 symmetry between particles and anti-particles is represented by the "-" sign in front of the time derivative.

The differential system that satisfies all these assumptions is as follows:

$$\begin{aligned}
-i\hbar\frac{\partial}{\partial t}\psi &= mc^2\psi - i\hbar c\sigma \cdot \nabla\bar{\varphi},\\
i\hbar\frac{\partial}{\partial t}\bar{\varphi} &= jmc^2\bar{\varphi} - i\hbar c\sigma \cdot \nabla\chi,\\
-i\hbar\frac{\partial}{\partial t}\chi &= j^2mc^2\chi - i\hbar c\sigma \cdot \nabla\bar{\psi},\\
i\hbar\frac{\partial}{\partial t}\bar{\psi} &= mc^2\bar{\psi} = -i\hbar c\sigma \cdot \nabla\varphi,\\
-i\hbar\frac{\partial}{\partial t}\varphi &= j^2mc^2\varphi - i\hbar c\sigma \cdot \nabla\bar{\chi},\\
i\hbar\frac{\partial}{\partial t}\bar{\chi} &= jmc^2\bar{\chi} - i\hbar c\sigma \cdot \nabla\psi,
\end{aligned} \tag{63}$$

Here we made a simplifying assumption that the mass operator is just proportional to the identity matrix, and therefore commutes with the operator $\sigma \cdot \nabla$. The functions ψ, φ and χ are related to their conjugates via the following third-order equations:

$$\begin{aligned}
-i\frac{\partial^3}{\partial t^3}\psi &= \left[\frac{m^3c^6}{\hbar^3} - i(\sigma\cdot\nabla)^3\right]\bar{\psi} = \left[\frac{m^3c^6}{\hbar^3} - i\sigma\cdot\nabla\right](\Delta\bar{\psi}),\\
i\frac{\partial^3}{\partial t^3}\bar{\psi} &= \left[\frac{m^3c^6}{\hbar^3} - i(\sigma\cdot\nabla)^3\right]\psi = \left[\frac{m^3c^6}{\hbar^3} - i\sigma\cdot\nabla\right](\Delta\psi),
\end{aligned} \tag{64}$$

and the same, of course, for the remaining wave functions φ and χ.

The overall $Z_2 \times Z_3$ symmetry can be grasped much better if we use the matrix notation, encoding the system of linear equations (63) as an operator acting on a single vector composed of all the components. Then the system (63) can be written with the help of the following 6×6 matrices composed of blocks of 3×3 matrices as follows:

$$\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}, \tag{65}$$

with I the 3×3 identity matrix, and the 3×3 matrices B_1, B_2 and $Q = Q_3$ defined in (53) and (??). follows:

The matrices B_1 and Q_3 generate the algebra of traceless 3×3 matrices with determinant 1, introduced by Sylvester and Cayley under the name of *nonionalgebra*. With this notation, our set of equations (63) can be written in a very compact way:

$$-i\hbar\Gamma^0 \frac{\partial}{\partial t} \Psi = [Bm - i\hbar Q\sigma \cdot \nabla] \Psi, \tag{66}$$

Here Ψ is a column vector containing the six fields, $[\psi, \varphi, \chi, \bar{\psi}, \bar{\varphi}, \bar{\chi}]$, in this order.

But the same set of equations can be obtained if we dispose the six fields in a 6×6 matrix, on which the operators in (66) act in a natural way:

$$\Psi = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix}, \quad \text{with} \quad X_1 = \begin{pmatrix} 0 & \psi & 0 \\ 0 & 0 & \phi \\ \chi & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & \bar{\chi} \\ \bar{\psi} & 0 & 0 \\ 0 & \bar{\varphi} & 0 \end{pmatrix} \tag{67}$$

By consecutive application of these operators we can separate the variables and find the common equation of sixth order that is satisfied by each of the components:

$$-\hbar^6 \frac{\partial^6}{\partial t^6} \psi - m^6 c^{12} \psi = -\hbar^6 \Delta^3 \psi. \tag{68}$$

Identifying quantum operators of energy and the momentum, $-i\hbar \frac{\partial}{\partial t} \to E, \quad -i\hbar\nabla \to \mathbf{p}$, we can write (68) simply as follows:

$$E^6 - m^6 c^{12} =\mid \mathbf{p} \mid^6 c^6. \tag{69}$$

This equation can be factorized showing how it was obtained by subsequent action of the operators of the system (63):

$$E^6 - m^6 c^{12} = (E^3 - m^3 c^6)(E^3 + m^3 c^6) =$$

$$(E - mc^2)(jE - mc^2)(j^2 E - mc^2)(E + mc^2)(jE + mc^2)(j^2 E + mc^2) =\mid \mathbf{p} \mid^6 c^6.$$

The equation (68) can be solved by separation of variables; the time-dependent and the space-dependent factors have the same structure:

$$A_1 \, e^{\omega t} + A_2 \, e^{j \omega t} + A_3 e^{j^2 \omega t}, \qquad B_1 \, e^{\mathbf{k}.\mathbf{r}} + B_2 \, e^{j \, \mathbf{k}.\mathbf{r}} + B_3 \, e^{j^2 \, \mathbf{k}.\mathbf{r}}$$

with ω and $\mathbf{k}$ satisfying the following dispersion relation:

$$\frac{\omega^6}{c^6} = \frac{m^6 c^6}{\hbar^6} + \mid \mathbf{k} \mid^6, \tag{70}$$

where we have identified $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$.

The relation 70) is invariant under the action of $Z_2 \times Z_3$ symmetry, because to any solution with given real ω and $\mathbf{k}$ one can add solutions with ω replaced by $j\omega$ or $j^2\omega$, $j\mathbf{k}$ or $j^2\mathbf{k}$, as well as $-\omega$; there is no need to introduce also $-\mathbf{k}$ instead of $\mathbf{k}$ because the vector $\mathbf{k}$ can take on all possible directions covering the unit sphere.

The nine complex solutions can be displayed in two 3×3 matrices as follows:

$$\begin{pmatrix} e^{\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{\omega t - j\mathbf{k}\cdot\mathbf{r}} & e^{\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}} & e^{j\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{j^2\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{j^2\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix}, \quad \begin{pmatrix} e^{-\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{-\omega t - j\mathbf{k}\cdot\mathbf{r}} & e^{-\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{-j\omega t - j\mathbf{k}\cdot\mathbf{r}} & e^{-j\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \\ e^{-j^2\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega t - \mathbf{k}\cdot\mathbf{r}} & e^{-j^2\omega t - j^2\mathbf{k}\cdot\mathbf{r}} \end{pmatrix}$$

and their nine independent products can be represented in a basis of real functions as

$$\begin{pmatrix} A_{11}\, e^{\omega t - \mathbf{k}\cdot\mathbf{r}} & A_{12}\, e^{\omega t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\mathbf{k}\cdot\xi) & A_{13}\, e^{\omega t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\mathbf{k}\cdot\xi) \\ A_{21}\, e^{-\frac{\omega t}{2} - \mathbf{k}\cdot\mathbf{r}} \cos\omega\tau & A_{22}\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\omega\tau - \mathbf{k}\cdot\xi) & A_{23}\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\omega\tau + \mathbf{k}\cdot\xi) \\ A_{31}\, e^{-\frac{\omega t}{2} - \mathbf{k}\cdot\mathbf{r}} \sin\omega\tau & A_{32}\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\omega\tau + \mathbf{k}\cdot\xi) & A_{33}\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\omega\tau - \mathbf{k}\cdot\xi) \end{pmatrix}$$

where $\tau = \frac{\sqrt{3}}{2}t$ and $\xi = \frac{\sqrt{3}}{2}\mathbf{kr}$; the same can be done with the conjugate solutions (with $-\omega$ instead of ω).

The functions displayed in the matrix do not represent a wave; however, one can produce a propagating solution by forming certain cubic combinations, e.g.

$$e^{\omega t - \mathbf{k}\cdot\mathbf{r}}\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\omega\tau - \mathbf{k}\cdot\xi)\, e^{-\frac{\omega t}{2} + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\omega\tau - \mathbf{k}\cdot\xi) = \frac{1}{2}\sin(2\omega\tau - 2\mathbf{k}\cdot\xi).$$

What we need now is a multiplication scheme that would define triple products of non-propagating solutions yielding propagating ones, like in the example given above, but under the condition that the factors belong to three distinct subsets b(which can be later on identified as "colors"). This can be achieved with the 3×3 matrices of three types, containing the solutions displayed in the matrix, distributed in a particular way, each of the three matrices containing the elements of one particular line of the matrix:

$$[A] = \begin{pmatrix} 0 & A_{12}\, e^{\omega t - \mathbf{k}\cdot\mathbf{r}} & 0 \\ 0 & 0 & A_{23}\, e^{\omega t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos\mathbf{k}\cdot\xi \\ A_{31} e^{\omega t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin\mathbf{k}\cdot\xi & 0 & 0 \end{pmatrix} \tag{71}$$

$$[B] = \begin{pmatrix} 0 & B_{12}\, e^{-\frac{\omega}{2}t + \frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\tau + \mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & B_{23}\, e^{-\frac{\omega}{2}t - \mathbf{k}\cdot\mathbf{r}} \sin\tau \\ B_{31} e^{\omega t - \mathbf{k}\cdot\mathbf{r}} \cos\tau & 0 & 0 \end{pmatrix} \tag{72}$$

$$[C] =$$

$$\begin{pmatrix} 0 & C_{12}\, e^{-\frac{\omega}{2}t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\tau+\mathbf{k}\cdot\xi) & 0 \\ 0 & 0 & C_{23}\, e^{-\frac{\omega}{2}t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}} \sin(\tau-\mathbf{k}\cdot\xi) \\ C_{31} e^{-\frac{\omega}{2}t+\frac{\mathbf{k}\cdot\mathbf{r}}{2}} \cos(\tau+\mathbf{k}\cdot\xi) & 0 & 0 \end{pmatrix} \quad (73)$$

Now it is easy to check that in the product of the above three matrices, ABC all real exponentials cancel, leaving the periodic functions of the argument $\tau + \mathbf{k}\cdot\mathbf{r}$. The trace of this triple product is equal to $Tr(ABC) =$

$$[\sin\tau\, \cos(\mathbf{k}\cdot\mathbf{r}) + \cos\tau\, \sin(\mathbf{k}\cdot\mathbf{r})]\, \cos(\tau+\mathbf{k}\cdot\mathbf{r}) + \cos(\tau+\mathbf{k}\cdot\mathbf{r})\, \sin(\tau+\mathbf{k}\cdot\mathbf{r}),$$

representing a plane wave propagating towards $-\mathbf{k}$. Similar solution can be obtained with the opposite direction. From four such solutions one can produce a propagating Dirac spinor.

This model makes free propagation of a single quark impossible, (except for a very short distances due to the damping factor), while three quarks can form a freely propagating state.

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